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On the Compatibility of Spinor Field Equations with Regard to Generalisations of Weyl's Lemma

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Abstract

Generalisations of Weyl's lemma are discussed. In order to secure the compatibility of the spinor field equations, the generalisations may not be arbitrary. It is shown that a contracted Weyl lemma must be valid. This lemma 'saves' the duality between the Lorentz covariant and the Einstein covariant representation of the equation of continuity. The meaning of Weyl's lemma and its generalisations is discussed in terms of the fibre bundle theory.

For the Lorentz covariant representation of spinor field equations and thus for the description of the influence of the gravitational field on the dynamics of Fermi particles, in general relativity the validity of Weyl's lemma is assumed; i.e. covariant constance of the metric spinors is postulated (Weyl, 1929). The supposition of Weyl's lemma enables the use of the formalism of fibre bundles for the representation of spinors in Riemann–Einstein spaces (Treder & Borzeszkowski, 1971).

The validity of Weyl's lemma is primarily a physical question, however; in fact, Weyl's lemma asserts that the tensorial quantities formed by fusion of spinors obey the transport laws for genuine tensors following from the covariance principle. From this we see that, physically speaking, Weyl's lemma says that the Bose particles formed by fusion of Fermi particles experience the same geometrisable universal forces as elementary (genuine) bosons. Corresponding to Weyl's lemma, Einstein's principle of equivalence holds both for the Fermi and for the Bose particles (Treder, 1971, 1972).

Therefore, it is physically reasonable to alter the general relativistic spinor calculus such that Weyl's lemma is no longer valid. Weyl himself has proposed such a modification in order to make the affine transport of fermions directly dependent on the structure of the matter fields. But the violation of Weyl's lemma makes it necessary to consider explicitly postulates being satisfied identically in case of the validity of Weyl's lemma. Especially, the meaning of the Lorentz covariant formulation of the spinor equations depends on the form of the special relativistic equations one starts from.

If Weyl's lemma is not valid for the metric spinors, i.e. if we have (Treder, 1972)

$$\begin{aligned}\sigma^{i}{}_{\mu\dot{\nu}\parallel l} &= \sigma^{i}{}_{\mu\dot{\nu},l} - \Lambda^{\alpha}{}_{\mu l}\,\sigma^{i}{}_{\alpha\dot{\nu}} - \Lambda^{\dot{\alpha}}{}_{\dot{\nu}l}\,\sigma^{i}{}_{\mu\dot{\alpha}} \\ &\equiv \Sigma^{i}{}_{\mu\dot{\nu}l} \neq 0\end{aligned}$$

then the Hermitian symmetry of the spinor field equations is no longer directly compatible with the rule: 'The Einsteinian gravitation field is coupled to Fermi fields by means of general covariant writing of the Weyl or the Dirac equations, respectively' (Treder, 1971).

Indeed, for example, the Lagrangian

$$L = \varphi^{\dot{\nu}}\,\sigma^{l}{}_{\mu\dot{\nu}}\,\varphi^{\mu}{}_{\parallel l} \tag{1}$$

is no longer equivalent to the complex conjugate Langrangian

$$L^{*} = \varphi^{\nu}\,\sigma^{l}{}_{\dot{\mu}\nu}\,\varphi^{\dot{\mu}}{}_{\parallel l} \tag{1a}$$

if Weyl's lemma is not valid; in this case L and L^{*} are no longer Hermitian symmetric up to a four-divergence. Instead, we have

$$\varphi^{\dot{\nu}}\,\sigma^{l}{}_{\mu\dot{\nu}}\,\varphi^{\mu}{}_{\parallel l} - \tfrac{1}{2}(\varphi^{\dot{\nu}}\,\varphi^{\mu}\,\sigma^{l}{}_{\mu\dot{\nu}})_{;l} = \tfrac{1}{2}(\varphi^{\dot{\nu}}\sigma^{l}{}_{\mu\dot{\nu}}\,\varphi^{\mu}{}_{\parallel l} - \varphi^{\mu}\,\sigma^{l}{}_{\mu\dot{\nu}}\,\varphi^{\dot{\nu}}{}_{\parallel l}) - \tfrac{1}{2}\varphi^{\mu}\,\varphi^{\dot{\nu}}\,\sigma^{l}{}_{\mu\dot{\nu}\parallel l} \tag{2}$$

From (1) one obtains the equations

$$\frac{\delta L}{\delta\varphi^{\mu}} = -(\varphi^{\dot{\nu}}\,\sigma^{l}{}_{\mu\dot{\nu}})_{\parallel l} = 0 \tag{3a}$$

and

$$\frac{\delta L}{\delta\varphi^{\dot{\nu}}} = \sigma^{l}{}_{\mu\dot{\nu}}\,\varphi^{\mu}{}_{\parallel l} = 0 \tag{3b}$$

being only compatible if the contracted Weyl lemma given by

$$\sigma^{l}{}_{\mu\dot{\nu}\parallel l} = \Sigma^{l}{}_{\mu\dot{\nu}l} = 0 \tag{4}$$

is valid. Assuming the validity of (4), (1) and (1a) are equivalent because they are Hermitian up to a divergence.

On the other hand, when starting from the Hermitian symmetric Lagrangian (Wentzel, 1949),

$$\begin{aligned}\bar{L} &= \tfrac{1}{2}(L + L^{*}) \\ &= \tfrac{1}{2}(\varphi^{\dot{\nu}}\,\sigma^{l}{}_{\mu\dot{\nu}}\,\varphi^{\mu}{}_{\parallel l} - \varphi^{\mu}\,\sigma^{l}{}_{\mu\dot{\nu}}\,\varphi^{\dot{\nu}}{}_{\parallel l})\end{aligned} \tag{5}$$

we no longer obtain the covariant Weyl equation, but instead of the Weyl equation we obtain the equations

$$\frac{\delta\bar{L}}{\delta\varphi^{\mu}} = -\sigma^{l}{}_{\mu\dot{\nu}}\,\varphi^{\dot{\nu}}{}_{\parallel l} - \tfrac{1}{2}\sigma^{l}{}_{\mu\dot{\nu}\parallel l}\,\varphi^{\dot{\nu}} = 0 \tag{6a}$$

and

$$\frac{\delta \bar{L}}{\delta \varphi^{\dot{\nu}}} = \sigma^{l}{}_{\mu\dot{\nu}}\, \varphi^{\mu}{}_{\parallel l} + \tfrac{1}{2}\sigma^{l}{}_{\mu\dot{\nu}\parallel l}\, \varphi^{\mu} = 0 \tag{6b}$$

Only if the contracted Weyl lemma (4) is valid, (6a) and (6b) are again identical with the covariant Weyl equations.

Let us consider now, according to Weyl's proposal, the spinor affinities $\Lambda^{\alpha}{}_{\beta l}$ and $\Lambda\varphi^{\dot{\alpha}}_{\dot{\beta} l}$ as independent field functions appearing additionally to the metric and the metric spinors $\sigma^{l}{}_{\mu\dot{\nu}}(x^{i})$. Then the Hermitian symmetric general relativistic Lagrangian density

$$\mathscr{L}_{\text{rel}} = \sqrt{(-g)}\left(-R + \frac{i\kappa}{2}\,[\varphi^{\dot{\nu}}\,\sigma^{l}{}_{\mu\dot{\nu}}\,\varphi^{\mu}{}_{\parallel l} - \varphi^{\mu}\,\sigma^{l}{}_{\mu\dot{\nu}}\,\varphi^{\dot{\nu}}{}_{\parallel l}]\right) \tag{7}$$

(R is the Riemannian curvature scalar) provides spinor affinities which satisfy the contracted Weyl lemma (4). Indeed, for $\Lambda^{\mu}{}_{\alpha k}$ and $\Lambda^{\dot{\mu}}{}_{\dot{\alpha} k}$, respectively, follow the field equations (Weyl, 1950):

$$\frac{1}{\sqrt{-g}}\frac{\delta \mathscr{R}}{\delta \Lambda} = \frac{i\kappa}{2}\,\sigma^{k}{}_{\mu\dot{\nu}}\,\varphi^{\alpha}\,\varphi^{\dot{\nu}} \tag{8a}$$

and

$$\frac{1}{\sqrt{-g}}\frac{\delta \mathscr{R}}{\delta \Lambda^{*}} = -\frac{i\kappa}{2}\,\sigma^{k}{}_{\nu\dot{\mu}}\,\varphi^{\dot{\alpha}}\,\varphi^{\nu} \tag{8b}$$

respectively. On the other hand, starting from (1) and (1a), i.e. starting from the Lagrangian densities,

$$\mathscr{L}_{\text{rel}} = \sqrt{(-g)}\,(-R + i\kappa\varphi^{\dot{\nu}}\,\sigma^{l}{}_{\mu\dot{\nu}}\,\varphi^{\mu}{}_{\parallel l}) \tag{9a}$$

and

$$\mathscr{L}^{*}{}_{\text{rel}} = \sqrt{(-g)}\,(-R - i\kappa\varphi^{\nu}\,\sigma^{l}{}_{\dot{\mu}\nu}\,\varphi^{\dot{\mu}}{}_{\parallel l}) \tag{9b}$$

respectively, the field equations

$$\frac{1}{\sqrt{-g}}\frac{\delta \mathscr{R}}{\Lambda \delta} = i\kappa\sigma^{k}{}_{\mu\dot{\nu}}\,\varphi^{\alpha}\,\varphi^{\dot{\nu}} \tag{10a}$$

and

$$\frac{1}{\sqrt{-g}}\frac{\delta \mathscr{R}}{\delta \Lambda^{*}} = -i\kappa\sigma^{k}{}_{\nu\dot{\mu}}\,\varphi^{\dot{\alpha}}\,\varphi^{\nu} \tag{10b}$$

result.

It is seen that the vanishing terms on the right sides of (8) and (10), respectively, corresponds with the validity of Weyl's lemma. However, the contracted lemma (4) is valid for $\Lambda^{\mu}{}_{\nu l}$ and $\Lambda^{\dot{\mu}}{}_{\dot{\nu} l}$ in consequence of the field equations (8) and (10):

$$\begin{aligned}\Sigma^{l}{}_{\mu\dot{\nu} l} &= \text{const.}\,(\sigma^{l}{}_{\mu\dot{\alpha}}\,\sigma_{l\beta\dot{\nu}}\,\varphi^{\beta}\,\varphi^{\dot{\alpha}} - \sigma^{l}{}_{\alpha\dot{\nu}}\,\sigma_{l\mu\dot{\beta}}\,\varphi^{\alpha}\,\varphi^{\dot{\beta}})\\ &= \text{const.}\,(\gamma_{\mu\beta}\,\gamma_{\dot{\alpha}\dot{\nu}}\,\varphi^{\beta}\,\varphi^{\dot{\alpha}} - \gamma_{\alpha\mu}\,\gamma_{\dot{\nu}\dot{\beta}}\,\varphi^{\alpha}\,\varphi^{\dot{\beta}}) = 0\end{aligned} \tag{11}$$

($\gamma_{\mu\beta}$ is the metric tensor of the spinor space.) By this (10a) and (10b) become compatible with each other and become equivalent to the Lagrangian (7).

In addition we mention that, since generally the general spinor transport is not metrical, it makes a difference whether one writes the Lagrangian in the form (7) or as

$$L_{\text{rel}} = -R + \frac{i\kappa}{2}(\varphi_{\dot{\nu}}\sigma_k{}^{\mu\dot{\nu}}\varphi_{\mu\,\|\,l} - \varphi_\mu\sigma_k{}^{\mu\dot{\nu}}\varphi_{\dot{\nu}\,\|\,l})g^{kl} \tag{12}$$

From (12) it follows

$$\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{R}}{\delta\Lambda} = \frac{i\kappa}{2}\sigma_l{}^{\mu\dot{\beta}}\varphi_{\dot{\beta}}\varphi_\alpha g^{kl} \tag{13a}$$

and

$$\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{R}}{\delta\Lambda^*} = -\frac{i\kappa}{2}\sigma_l{}^{\beta\dot{\mu}}\varphi_\beta\varphi_{\dot{\alpha}}g^{kl} \tag{13b}$$

respectively. The contracted Weyl lemma follows from (13a, b), too.

In order to discuss the generalisation (4) of Weyl's lemma in more detail, we consider a general mathematical way of looking at the meaning of Weyl's lemma.

For this end, we turn to the investigation of the transport of tensorial quantities formed by fusion of spinors. We write

$$\sigma^i{}_{\mu\dot{\nu}\,\|\,l} = \Sigma^i{}_{\mu\dot{\nu}l} \tag{14}$$

where $\sigma^i{}_{\mu\dot{\nu}}$ is given by $\sigma^i{}_{\mu\dot{\nu}} = h^i{}_A\sigma^A{}_{\mu\dot{\nu}}$ ($\sigma^A{}_{\mu\dot{\nu}}$ are the Pauli-spin-matrices). We now treat the index combination $(\mu\dot{\nu})$ as one index with respect to the transport, also. Therefore, we postulate the covariant constancy of the Pauli-spin-matrices:

$$\sigma^A{}_{\mu\dot{\nu}\,\|\,l} = \sigma_A{}^{\mu\dot{\nu}}{}_{\|\,l} = 0 \tag{15}$$

This lemma connects the Lorentz covariant derivatives with the spinor transport. We have

$$\begin{aligned}(h^i{}_A\sigma^A{}_{\mu\dot{\nu}})_{\|\,l} = \sigma^A{}_{\mu\dot{\nu}}h^i{}_{A\,\|\,l} &= \Sigma^i{}_{\mu\dot{\nu}l}\\ &= \sigma^A{}_{\mu\dot{\nu}}\Sigma^i{}_{Al}\end{aligned} \tag{16}$$

where

$$h^i{}_{A\,\|\,l} = \Sigma^i{}_{Al}$$

The meaning of Weyl's lemma written in the form (Treder, 1972)

$$h^i{}_{A\,\|\,l} = 0$$

is discussed now in terms of the mathematics of fibre bundle theory. We start by considering the tangent bundle $T(V_4)$ over V_4. $T(V_4)$ is the bundle associated with $L(V_4)$ with standard fibre $\mathbf{R}^n$ where $L(V_4)$ is the principal fibre bundle of linear frames.†

Now it is known how a connection can be defined in a vector bundle.‡ In the special case of a tangent bundle $T(V_4)$ a connection Γ is called a linear connection over the manifold V_4.

† $\mathbf{R}^n$ is the vector space of all 4-tuples $(\xi^1, \ldots, \xi^4)$; $\xi^1, \ldots, \xi^4$ are real numbers.
‡ See, for example, Kobayashi & Nomizu (1963).

To the transformation $(x^i, X^A{}_i) \rightarrow (x^{i'}, X^{A'}_{i'})$ of the bundle coordinates there corresponds the following transformation of the local components L^A_{Bi} of the connection Γ:

$$L^{A'}_{B'i'} = \frac{\partial x^i}{\partial x^{i'}} \omega^{A'}_A \omega^B_{B'} L^A_{Bi} + \omega^{A'}_A \omega^A_{B',i'}. \tag{17}$$

Here $\partial x^i/\partial x^{i'}$ corresponds to the transformation $x^i \rightarrow x^{i'}$ and $\omega^{A'}_A$ (with $\omega^{A'}_A \omega^A_{B'} = \delta^{A'}_{B'}$) to the transformation $X^A_i \rightarrow X^{A'}_i$.

Using in $T(V_4)$ natural bundle coordinates X^k_i one obtains

$$\omega^{A'}_A = \omega^{a'}_a = \partial x^{a'}/\partial x^a$$

and (17) reduces to the transformation law of a linear connection over V_4 under coordinate transformations:

$$\Gamma^{i'}_{k'l'} = \frac{\partial x^l}{\partial x^{l'}} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^{i'}}{\partial x^i} \Gamma^i_{kl} + \frac{\partial x^{i'}}{\partial x^i} \frac{\partial^2 x^i}{\partial x^{k'} \partial x^{l'}} \tag{18}$$

On the other hand, for $x^i = \delta^i_{i'} x^{i'}$ from (17) one obtains the transformation law of the Lorentz connection:

$$L^{A'}_{B'l} = \omega^{A'}_A \omega^B_{B'} L^A_{Bl} + \omega^{A'}_A \omega^A_{B',l} \tag{19}$$

Let us now consider a transformation $X^A_i = h^A_k X^k_i$ (with $h^A_i \in GL(4, \mathbf{R})$) of the bundle coordinates. According to (17), then the following transformation $\Gamma^i_{kl} \rightarrow \Gamma^A_{Bl}$ of the local components of Γ is associated with it†:

$$\begin{aligned} L^A_{Bl} &= h^A_k h^m_B L^k_{ml} + h^A_m h^m_{B,l} \\ &= h^A_m h^m_{B;l} \end{aligned} \tag{20}$$

This means that L^A_{Bl} and Γ^k_{ml} are local components of the connection Γ only with respect to different bundle coordinates. From (20) one gets the components Γ^k_{ml} expressed by L^A_{Bl}, h^A_m, and $h^m_{B,l}$,

$$\Gamma^k_{ml} = h^k_A h^B_m L^A_{Bl} - h^B_m h^k_{B,l} \tag{21}$$

By multiplication of (20) by h^r_A it follows from (20)

$$h^r_{B\parallel l} \equiv h^r_{B,l} + \Gamma^r_{ml} h^m_B - L^A_{Bl} h^r_A = 0$$

Defining for mixed quantities T_{Al} the general covariant derivative

$$T_{Al\parallel i} = T_{Al,i} - \Gamma^k_{li} T_{Ak} - L^B_{Ai} T_{Bl} \tag{22}$$

where Γ^k_{li} and L^B_{Ai} are the local components of one and the same connection Γ of $T(V_4)$, Weyl's lemma is automatically fulfilled; it is identical with the transformation law (20) and (21), respectively.

Let us assume now that, instead of Weyl's lemma, the relation

$$h^i_{A\parallel l} = \Sigma^i_{Al} \neq 0 \tag{23}$$

holds. This means that two different linear connections Γ and $\bar{\Gamma}$ are defined in the bundle; the derivative in (23) is formed by means of the components

† The formulae (18)–(20) follow from the general transformation law of Γ given in Kobayashi & Nomizu (1963), for example, by fixation of the bundle coordinates.

$(\Gamma^i_{kl}, \bar{L}^A_{Bl})$ or $(\bar{\Gamma}^i_{kl}, L^A_{Bl})$. On the other hand, defining the derivative in (23), as suggested by the fibre bundle formalism, by means of $(\Gamma^i_{kl}, L^A_{Bl})$ or $(\bar{\Gamma}^i_{kl}, \bar{L}^A_{Bl})$ Weyl's lemma is again valid, as we have seen above.

But starting from (23) and assuming, accordingly, that the derivative in (23) is formed by means of Γ^i_{kl} and $\bar{L}^A_{Bl}$, i.e., starting from

$$h^i_{A\parallel l} \equiv h^i_{A,l} + \Gamma^i_{ml} h^m_A - \bar{L}^B_{Al} h^i_B = \Sigma^i_{Al} \tag{23a}$$

one obtains, instead of (20),

$$\bar{\Delta}^c_{Al} = \Gamma^i_{ml} h^m_A h^c_i + h^i_{A,l} h^c_i - \Sigma^i_{Al} h^c_i \tag{24}$$

From (24) it follows by virtue of (21):

$$\bar{\Gamma}^i_{kl} = \Gamma^i_{kl} - \Sigma^i_{Al} h^A_k \tag{25}$$

Assuming now that

$$\Gamma^i_{kl} = \left\{ {i \atop kl} \right\} = \tfrac{1}{2} g^{im}(g_{mk,l} + g_{lm,k} - g_{kl,m})$$

we can say that Ricci's lemma does not hold for the derivative '$;l$' which is formed by means of $\bar{\Gamma}^i_{kl}$. Accordingly, the contracted Weyl lemma can be interpreted as follows.

According to Weyl, the derivatives '$\parallel l$' appearing in spinor equations are to be defined by means of $(\left\{ {i \atop kl} \right\}, \bar{L}^A_{Bl})$; especially, here $\left\{ {i \atop kl} \right\}$ determines the transport of genuine tensors. But to obtain a general duality between the Lorentz covariant and the coordinate covariant representation of tensorial quantities formed by spinors, one ought to define the derivatives through $(\bar{\Gamma}^i_{kl}, \bar{L}^A_{Bl})$; only then is Weyl's lemma (being necessary for this duality) satisfied. But, in this case, the non-genuine tensors which are formed by fusion of spinors are transported in a different way than genuine tensors; for instance, it yields

$$\begin{aligned} T^l{}_{;i} &= T^l{}_{,i} + T^k \bar{\Gamma}^l{}_{ki} \\ &= T^l{}_{;i} - \Sigma^l{}_{Ai} h^A{}_k T^k \end{aligned} \tag{26}$$

Now we have seen above that, for reasons of compatibility, the contracted Weyl lemma

$$\sigma^l_{\mu\dot{\nu}\parallel l} = \Sigma^l_{\mu\dot{\nu}l} = 0 \tag{27a}$$

and

$$h^l_{A\parallel l} = \Sigma^l_{Al} = 0 \tag{27b}$$

respectively, must be valid in a spinor field theory. With (27b) and (25) it follows that

$$\bar{\Gamma}^l_{kl} = \left\{ {l \atop kl} \right\}$$

and hence we obtain

$$T^i{}_{;i} = T^i{}_{;i} \tag{28}$$

From (28) follows that the continuity equation†

$$(\varphi^{\dot{\nu}} \sigma^l{}_{\mu\dot{\nu}} \varphi^\mu)_{\parallel l} = (\varphi^{\dot{\nu}} \sigma^l{}_{\mu\dot{\nu}} \varphi^\mu)_{;l} = 0 \tag{29}$$

† The derivatives in (3a), (3b), and thus in (29), are defined by (23), according to Weyl.

resulting (3a) and (3b) is also valid with regard to the derivative by means of $\bar{\Gamma}^i_{kl}$,

$$(\varphi^{\dot{\nu}} \sigma^l{}_{\mu\dot{\nu}} \varphi^\mu)_{;l} = 0 \qquad (30)$$

This means that the contracted Weyl lemma (27) just 'saves' the duality between the Lorentz covariant and the Einsteinian coordinate covariant representation of the equation of continuity following by fusion from the Weyl spinor field equations. Asking now this duality for all tensor equations formed by fusion one is again led to Weyl's lemma itself; then it is impossible to distinguish between 'genuine' and 'fusion' matter.

Indeed, only demanding the contracted Weyl lemma we have in general

$$T^{ik}{}_{;k} \neq 0 \qquad (31)$$

and

$$T^{ik}{}_{;k} \neq T^{ik}{}_{;k} \qquad (32)$$

From (31) and (32) it follows that, in case of the violation of Weyl's lemma, the principle of equivalence holds only for one sort of matter. According to Weyl, one could tend to abandon the equivalence principle for spinorial matter because, for this matter (additional to the gravitation), a further universal interaction could exist depending on the spinorial matter only.

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